Wave functions, momentum and the probability density of portfolio returns

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Abstract

Stock market prices are often modeled in terms of a particle that evolves as a Wiener process and the Fokker-Planck equation is then used to determine the distributional properties of the returns process. Yet numerous studies have found the Wiener process to provide a problematic basis for modelling stock market returns. We take the alternative approach of modelling stock market returns in terms of a particle that evolves in a potential energy well. The Schrödinger equation is then used to determine the wave function for the returns process. Our empirical analysis shows that the probability distributions implied by this model are strongly compatible with the returns earned by the individual components of the London Stock Exchange’s FTSE All Share Index over the period from 1994 until 2007. We also use the Schrödinger equation to determine the wave functions of investment portfolios and demonstrate how portfolio wave functions based on returns data for the FTSE All Share Index are strongly compatible with the given portfolio’s returns. Finally, we show both analytically and empirically that there is an inverse (conjugate) relationship between the variance associated with the momentum in portfolio returns and the variance of the portfolio returns process itself.

Key Words: econophysics; momentum; probability density; quantum tunneling; wave function

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1. Introduction

A commonly used procedure for gauging the distributional properties of stock market returns uses the Fokker-Planck equation in conjunction with a stochastic differential equation to describe how increments in the returns process evolve over time [1,2,3]. Under this model the instantaneous return on the stock market is viewed in terms of a particle that randomly evolves across the real line [4,5,6]. In particular, the particle can move a given distance to the right or left in any given period and its position on the line gives the return on the stock market during the affected epoch. The simplest interpretation of this model takes the particle to be “free” in which case there is an equal probability ($\frac{1}{2}$) of the particle moving to the right or left from its current position. The solution of the Fokker-Planck equation under this specification shows that the probability density for the position of the particle (correspondingly, the stock market return) to be the Gaussian distribution [4,6] from which much of the received theory of finance has evolved [5]. But there are more complicated and potentially more realistic processes through which to model the stock market returns process. One might, for example, consider an elastically bound particle under which the probability of the particle moving to the right or left hinges on the particle’s current position relative to the origin (the origin representing the long run mean or average return on the stock market). The farther the particle is removed to the right (left) of the origin the greater the probability of a movement to the left (right) during the next epoch. The solution of the Fokker-Planck equation under this specification shows that the probability density for the position of the particle (or equivalently, the stock market return) to be Gaussian but with a mean and variance that decay exponentially with time [4,6].

The principal difficulty with models based on the Fokker-Planck equation is that increments in the state variable are described in terms of a Wiener process. Here it is well known that the distributional properties of the Wiener process are completely characterised by its first two moments [6,7]. However, if the returns process encompasses “rare events” then its distributional properties will also be characterised by the higher moments which sample the tails of the given probability density [8]. Given this it is perhaps not surprising that numerous studies have found the Wiener process to provide a problematic basis for the modelling of stock market returns [1,2]. The Black
and Scholes [5,9] options pricing model for example is based on the assumption that the logarithm of the stock’s price evolves in terms of a Wiener process with known drift. Yet despite its elegance the Black and Scholes [9] model provides a generally poor description of the way option prices evolve over time [10]. An important point here, however, is that there is a well known relationship between the Fokker-Planck equation and the Schrödinger equation for certain potentials [11,12]. This in turn means that for particular potentials, the solution of the Fokker-Planck equation has the alternative interpretation of providing the probability amplitude (rather than the probability density) for the particle’s position. Given this, our purpose here is to use the Schrödinger equation to model the evolution of stock market returns in terms of a particle moving in a potential energy well but where there is a non-trivial probability of the particle tunneling into the well’s retaining walls. Whilst the transition probabilities for a particle moving in a potential well normally encompass the higher moments [13,14,15] and are generally more complex than those for the Weiner process [4,6], yet the formal analogy of the Fokker-Planck equation with the Schrödinger equation shows that the principles behind the two models are the same; namely, that the instantaneous return on the stock market is characterised by the position the particle occupies under the given potential.

Moreover, formalising the evolution of portfolio returns in terms of a particle moving in a potential well has the additional advantage of allowing one to develop a rigorous analysis of the stock returns momentum phenomenon that in recent years, has received considerable attention from empirical researchers. We note in particular that there is a considerable volume of empirical evidence which suggests that forming portfolios comprised of shares that have exhibited short term positive abnormal returns and selling shares exhibiting short term negative abnormal returns leads to significant medium term positive abnormal returns on the affected portfolios [16,17,18,19]. We use the Schrödinger equation to provide a formal definition of the returns momentum phenomenon and then using this definition show that the uncertainty associated with a share’s return momentum is inversely related to the uncertainty associated with the returns process itself. In other words, we show both analytically and empirically that the return on a share and its associated momentum are conjugate variables in the sense that the greater the variance of the return on a share, the less the variance associated with its return momentum and vice versa.
The next section briefly summarises the potential energy well and quantum tunneling model on which our analysis is based and then uses returns data covering the fourteen year period from 1994 until 2007 to estimate wave functions for the 31 components of the London Stock Exchange’s FTSE All Share Index. Our empirical analysis demonstrates how the wave functions for all but one of the 31 components of the FTSE All Share Index are strongly compatible with the returns data of the given components. In Section 3 we use the Schrödinger equation to determine the wave functions of investment portfolios and demonstrate how portfolio wave functions based on returns data for the FTSE All Share Index are strongly compatible with the given portfolio’s returns. In Section 4 we go on to show how an inverse (conjugate) relationship exists between the variance of the return on a share and the variance associated with its return momentum. The final section contains our summary conclusions.

2. Stock market return as a one-dimensional motion in a square potential well

We begin our analysis by defining \(x\) to be the instantaneous return of a particular asset listed on the stock exchange. The instantaneous return on the given asset is modelled in terms of a particle moving within a square potential energy well. It then follows that the wave function for the given asset will satisfy the Schrödinger equation [20,21,22]:

\[
\frac{\partial^2 \Psi(x,t)}{\partial x^2} + i\eta \frac{\partial \Psi(x,t)}{\partial t} - \xi \Psi(x,t) = 0
\]  

(1)

where \(x\) represents the particle’s position within the well (correspondingly, the instantaneous returns on the asset), \(\eta\) and \(\xi\) are physical constants related to the potential and kinetic energies of the particle and \(i = \sqrt{-1}\) is the imaginary unit. Now suppose one applies the substitution \(\Psi(x,t) = e^{-i\xi t} \psi(x)\) to equation (1) in which case it follows [23]:

\[
\frac{d^2 \psi(x)}{dx^2} = E \psi(x)
\]  

(2)
is the time-independent (or steady state) Schrödinger equation and $E = \xi (1 - \eta)$ is the energy eigenvalue associated with the given asset. Here it can be shown that wave functions with large energy eigenvalues (in absolute terms) lead to relatively “compact” probability densities for the position of the particle and thus, to comparatively small probabilities of “extreme” returns.

Now suppose the walls of the potential energy well are placed at the points $x = a$ and $x = b$ as depicted in Figure 1. Moreover, the particle’s kinetic energy is not sufficient for the particle to escape from the well. It then follows that region II constitutes the interior section of the well, although there is a non-trivial probability that the particle will penetrate through the walls into regions I and III of the well. However, if it the particle does penetrate into regions I and III it will eventually be reflected and is never absorbed into the walls of the well [21]. Furthermore, suppose region III is comprised of more porous material than the material comprising region I. This means the particle is likely to tunnel further into region III than it will into region I. It is not hard to show that the particle’s wave function will take the form [20]:

\[
\psi(x) = \begin{cases} 
C e^{\sqrt{\theta}(x-a)} & x \leq a \\
C \left[ \sqrt{\lambda \theta} \sin(\sqrt{\lambda}(x-a)) + \cos(\sqrt{\lambda}(x-a)) \right] & a < x \leq b \\
C \left[ \sqrt{\frac{\lambda}{\rho'}} \sin(\sqrt{\lambda}(b-a)) - \sqrt{\frac{\theta}{\rho'}} \cos(\sqrt{\lambda}(b-a)) \right] e^{-\sqrt{\rho'}(x-b)} & b < x 
\end{cases}
\]
where $\theta$ is the energy eigenvalue arising in region I of the well, $\lambda$ is the energy eigenvalue arising in region II of the well, $\gamma$ is the energy eigenvalue arising in region III of the well, $x(t) = \frac{P'(t)}{P(t)}$ is the instantaneous annualised return on the asset, $P(t)$ is the market value of the asset at time $t$, and:

$$C^2 = \left[ \frac{1}{2\sqrt{\theta}} + \frac{\lambda + \theta}{4\sqrt{\gamma}} + \frac{\lambda - \theta}{2\lambda} \cdot \sin(2\sqrt{\lambda}(b - a)) + \frac{\theta}{2\lambda}(1 - \cos(2\sqrt{\lambda}(b - a))) + \frac{(\theta - \lambda)}{4\sqrt{\gamma}} \cdot \cos(2\sqrt{\lambda}(b - a)) \right]^{-1}\cos(2\sqrt{\lambda}(b - a)) + \frac{\theta}{4\sqrt{\gamma}} \cdot \sin(2\sqrt{\lambda}(b - a))$$

The tunnelling points for the wave function, which (as previously noted) are $a$ and $b$, will satisfy the relationship [20]:

$$(b - a) = \frac{1}{\sqrt{\lambda}} \cdot \tan^{-1}(\sqrt{\frac{\lambda}{\theta}} + \sqrt{\frac{\theta}{\gamma}})$$

and mark the boundary points between the “regular” and “irregular” returns for the given asset.

Note how the wave function (4) given here implies that the tails of the probability density, $\psi^2(x)$, decay exponentially beyond the walls of the well. And as previously noted, the greater the energy eigenvalues, $E = \theta, \gamma$ (see Figure 1) the quicker the tails will decay. There is an important strand of literature [25,26,27] which suggests that stock market returns are characterised by probability distributions with fatter tails than can be accommodated by the exponential decay given in the above wave function. Against this [2,28,29] argue that exponential
decay characterises the tails of all but the very highest frequency returns for stocks comprising the Dow-Jones Industrial Average in the United States and that stock market returns in India, Japan, Germany and Brazil have probability distributions with tails that are more compatible with exponential decay than with any of the other specifications which have been suggested in the literature. Moreover, previous evidence [20] shows that the daily returns of the United Kingdom’s FTSE All Share Index are highly compatible with a probability distribution which exhibits exponential decay in the tails. Given that a significant strand of empirical evidence is strongly compatible with the hypothesis that stock market returns in many major industrialised countries are characterised by probability distributions with tails that exhibit exponential decay we now employ daily returns for the 31 components of the FTSE All Share Index over the period from 1994 to 2007 to estimate wave functions for the each component.

We begin our analysis by supposing that one has the ordered continuously compounded (logarithmic) return series, \( x_1 \leq x_2 \leq x_3 \leq \ldots \leq x_n \), for a particular component of the FTSE All Share Index. Then one may compute the simple average of the returns, \( \bar{x} = \frac{1}{n} \sum_{j=1}^{n} x_j \), and subtract it from each individual return to form the ordered centred return series

\[
\begin{align*}
  r_1 &= x_1 - \bar{x} \\
  r_2 &= x_2 - \bar{x} \\
  r_3 &= x_3 - \bar{x} \leq \ldots \leq r_n &= x_n - \bar{x}.
\end{align*}
\]

The ordered centred returns can then be used in conjunction with the Cramér-von Mises goodness of fit statistic [30]:

\[
T_3 = \frac{1}{12n} + \sum_{j=1}^{n} \left[ F(r_j) - \frac{j - \frac{1}{2}}{n} \right]^2
\]

where \( F(r_j) = \int_{-\infty}^{r_j} \psi^2(x)dx \) is the accumulated area under the probability density below the \( j^{th} \) ordered centred return, to estimate the parameters of the wave function (4). The Newton-Raphson algorithm [31] was then used to determine the values of the parameters of the wave function (4) that minimise the Cramér-von Mises statistic across the \( n = 2,816 \) daily centred returns available over the fourteen year period from 1 January, 1994 until 30 June, 2007 for each of the 31 components of the FTSE All Share Index on
which our analysis is based. The results of this exercise are summarised in Table 1.

Thus, the following estimates were obtained for the Aerodefence component of the FTSE All Share Index: \( \hat{\lambda} = 0.0484, \hat{\gamma} = 0.0229, \hat{\theta} = 0.0213, \hat{C} = 0.2713 \) and so on. Hence, our estimates suggest that regular centred returns for the Aerodefence component of the FTSE All Share Index vary from a low of \( \hat{a} = -2.4805 \) (or about –0.68 of 1% per day) up to \( \hat{b} = 2.9188 \) (or about 0.80 of 1% per day). Irregular returns are earned outside this interval. Figure 2 plots both the wave function, \( \psi(x) \), and the probability density function, \( \psi^2(x) \), for the Aerodefence daily centred return based on these estimated parameter values – both graphs being typical of the wave functions and probability densities for all thirty one components of the FTSE All Share Index.

That the estimated probability density provides a reasonable description of the way daily returns of the Aerodefence component of the FTSE All Share Index evolve in time is supported by the goodness of fit statistics summarised in the last two columns of Table 1. The first of these is the Cramér-von Mises test statistic, \( T_3 \). Anderson and Darling [32] have tabulated percentiles of the asymptotic distribution function of \( T_3 \) given that the centred returns, \( r_j \), on which \( T_3 \) is based are generated by the hypothesised probability distribution. These tables show that the probability of the \( T_3 \) exceeding 0.3473 is 10%, the probability of \( T_3 \) exceeding 0.4614 is 5% whilst the probability of \( T_3 \) exceeding 0.7435 is 1%. Hence, the return series on which our empirical analysis for the Aerodefence component of the FTSE All Share Index is based appears to be strongly
compatible with the hypothesised wave function (4) since the observed test statistic, 
\[ T_3 = 0.0712 \], is well below any of the generally accepted levels of significance for \( T_3 \) 
tabulated by Anderson and Darling [32]. However, a note of caution is in order here 
since \( T_3 \) is computed under the assumption that the hypothesised probability density 
function is completely specified [33]. Unfortunately, this is not the case in the present 
instance since there are four parameters – \( \lambda, \theta, \gamma \) and \( a \) – all of which have had to be 
estimated from the returns data for the Aerodefence component of the FTSE All Share 
Index.

Given this we also used the fitted probability density for the Aerodefence 
component of the FTSE All Share Index to compute the Cramér goodness of fit test 
statistic, \( T_0 \) [34]. This test statistic makes allowances for the fact that the hypothesised 
probability density may not have been completely specified; that is, some of the 
probability density’s parameters for the Aerodefence component have had to be estimated 
from the underlying data. The Cramér goodness of fit was implemented using the 
probability density, \( \psi_2(x) \), with the parameter estimates summarised earlier (that is, 
\( \hat{\lambda} = 0.0484, \hat{\gamma} = 0.0229, \hat{\theta} = 0.0213, \hat{C} = 0.2713 \) and so on) to determine the 5\(^{th}\), 10\(^{th}\), 
15\(^{th}\), ___ up to the 100\(^{th}\) percentile returns. We then used the actual number of returns 
falling between each of the \( n = 20 \) percentile returns as a basis for applying the Cramér 
goodness of fit test and thereby to determine the test statistic, \( T_0 \). If the return series is 
described by the wave function (4) then \( T_0 \) is asymptotically distributed as a Chi-square 
variate with \( n - k - 1 \) degrees of freedom, where \( k \) is the number of parameters estimated 
from the data [33]. Since \( k = 4 \) parameters are estimated from our returns data, this 
means \( T_0 \) is distributed with \( n - k - 1 = 20 - 4 - 1 = 15 \) degrees of freedom. Tabulated 
values of the Chi-square distribution with 15 degrees of freedom show that the 
probability of \( T_0 \) exceeding 22.31 is 10\%, the probability of \( T_0 \) exceeding 25.00 is 5\% 
whilst the probability of \( T_0 \) exceeding 30.58 is 1\%. Table I shows that the Cramér 
goodness of fit test statistic for the Aerodefence component turns out to be \( T_0 = 11.3283 \) 
which is both well below its expected value of \( E(T_0) = 15 \) and well within any of the 
generally accepted levels of significance for the Chi-square distribution. This again 
confirms that the return series for the Aerodefence component of the FTSE All Share
Index is strongly compatible with the hypothesised wave function (4) and the probability density implied by it.

The overwhelming weight of the empirical evidence summarised in Table 1 is that the wave function (4) does a good job in describing the evolution of daily returns for most components of the FTSE All Share Index. In general our parameter estimates appear to be sensible (e.g. the left hand limit, \(a\), of the potential well marking the boundary between “regular” and “irregular” returns is generally negative while the right hand limit, \(b\), is always positive). Moreover, the goodness of fit statistics, \(T_0\) and \(T_3\), are generally well within accepted levels of significance for the particular test. There is, however, one component of the FTSE All Share Index – Software and Computer Services – where there is evidence that the wave function (4) does not provide an adequate description of the way returns evolve in time. This component of the FTSE All Share Index returns a Cramér goodness of fit test statistic of \(T_0 = 140.29\) and this far exceeds the 1% level of significance for this test statistic, as given earlier. Moreover, the Cramér-von Mises goodness of fit test statistic, \(T_3 = 0.4222\), for this particular component of the FTSE All Share Index is also significant at the 10% level. It should be remembered, however, that at the 10% level of significance one would expect \((31 \times 0.1 \approx)\) two or perhaps three of the 31 components of the FTSE All Share Index to have goodness of fit statistics incompatible with the wave function (4) purely on the basis of chance. Hence, since Software and Computer Services is the only component of the FTSE All Share Index to return goodness of fit statistics which are incompatible with the wave function (4) and since this could have occurred purely by chance, we do not exclude the Software and Computer Services component of the FTSE All Share Index from our subsequent analysis.

3. Portfolio Formation

Consider a portfolio comprised of a single risky asset and a riskless asset with a known or sure return of \(q\) (per annum). Let \(\alpha\) be the proportionate investment in the risky asset in which case the instantaneous return on the portfolio will be \(z = q + (x - q)\alpha\) where, as previously, \(x\) is the instantaneous annual return on the risky asset. Now from previous analysis the return, \(x\), on the risky asset evolves in terms of the Schrödinger
equation (2). It then follows that the Schrödinger equation for the instantaneous return on the portfolio will be [35]:

\[ \frac{d^2 \psi(z)}{dz^2} = \frac{E}{\alpha^2} \psi(z) \]  

(6)

Note that as the proportionate investment, \( \alpha \), in the risky asset declines the energy eigenvalue, \( \frac{E}{\alpha^2} \), in the Schrödinger equation increases in magnitude. This in turn implies [15]:

\[ \text{Limit } \alpha \to 0 \quad \psi^2(z) = \delta(z - q) \]  

(7)

or that the probability density, \( \psi^2(z) \), for instantaneous return, \( z \), on the portfolio approaches a Dirac delta function, \( \delta(z - q) \), centred on the sure return, \( q \), as the proportionate investment, \( \alpha \), in the risky asset declines in magnitude. This result has the important consequence that wave functions with large energy eigenvalues (in absolute terms) lead to relatively “compact” probability densities and thus, to comparatively small probabilities of “extreme” returns. In other words, the relative magnitude of the eigenvalues associated with the returns on a given portfolio are a good measure of the risk associated with the portfolio’s return.

The eigenvalues of a portfolio comprised purely of risky assets will depend on the coherence levels between the wave functions of the assets comprising the portfolio [23]. Here our previously reported empirical analysis reveals considerable coherence between the wave functions of all 31 components of the FTSE All Share Index and so, portfolios formed from these components tend to involve constructive interference across the wave functions comprising the portfolio. The constructive interference will mean that the portfolio return has a comparatively small variance and so, realised portfolio returns will tend to be relatively close to the mean return. One can illustrate the importance of this point by again assuming portfolio returns evolve in terms of a particle moving within a square potential well analogous to that depicted in Figure 1. We also assume that an
equal proportionate investment, $\alpha_k = \frac{1}{31}$, is made in each of the $k = 1, 2, \ldots, m = 31$ components comprising the FTSE All Share Index. It then follows that the realised return on the portfolio will be $x = \sum_{k=1}^{31} \alpha_k x_k = \frac{1}{31} \sum_{k=1}^{31} x_k$, where $x_k$ is the return on the $k$th of the 31 components of the FTSE All Share Index.

Using the $n = 2816$ daily returns on an equally weighted portfolio of the 31 components of the FTSE All Share Index covering the fourteen year period from 1 January, 1994 until 30 June, 2007 shows that the minimised Cramér-von Mises test statistic for the wave function (4) is $T_3 = 0.0347$. Likewise, the Cramér goodness of fit statistic, which is asymptotically distributed as a Chi-Square variate with 15 degrees of freedom, turns out to be $T_0 = 8.4412$. Both these goodness of fit statistics are well within generally accepted levels of significance and thereby confirm that the returns for the equally weighted portfolio of assets are strongly compatible with the hypothesised wave function (4). Figure 3 plots both the wave function, $\psi(x)$, and the probability density function, $\psi^2(x)$, for the equally weighted portfolio’s centred returns and also, provides complete details of the estimated parameters on which the wave function is based. As previously noted constructive interference between the wave functions of the 31 components comprising the FTSE All Share Index means that the probability density for the portfolio return will tend to be much more compact than the probability densities of the individual components comprising the portfolio. Hence, it is hardly surprising that the probability density for the portfolio centred return, as depicted in Figure 3, is much more compact than the probability density for the centred returns of the Aerodefence industry depicted in Figure 2 – which, as we have previously noted, is typical of the probability densities for all thirty one components of the FTSE All Share Index.
It is highly unlikely, of course, that investors will want to establish portfolios with equal proportionate investments in the assets available to them. Given this, consider an investor who wishes to form a portfolio with the maximum possible diversification but subject to an expected return threshold of $\mu$. In other words, the investor wishes to minimize $\sum_{k=1}^{m} \alpha_k^2$ (thereby maximising the portfolio’s diversification) subject to the requirements $\sum_{k=1}^{m} \alpha_k = 1$ (all wealth is invested) and $\mu = \sum_{k=1}^{m} \alpha_k E(x_k)$ where $E(x_k)$ is the expected return on the $k^{th}$ component of the portfolio (the expected return threshold). One can determine the solution to this constrained minimisation problem using standard methods and thereby show that the proportionate investment in the $k^{th}$ component of the portfolio will be (see the Appendix) [36]:

$$\alpha_k = \frac{\lambda_2 E(x_k) + \lambda_1}{2}$$

(8)

where:

$$\lambda_1 = \frac{2 - \lambda_2 \sum_{k=1}^{m} E(x_k)}{m}$$

(9a)

and

$$\lambda_2 = \frac{2}{m} \cdot \frac{\mu - \frac{1}{m} \sum_{k=1}^{m} E(x_k)}{\frac{1}{m} \sum_{k=1}^{m} [E(x_k)]^2 - \frac{1}{m^2} [\sum_{k=1}^{m} E(x_k)]^2}$$

(9b)

Note that the denominator of the second term on the right hand side for $\lambda_2$ in equation (9b) is the variance of the expected return across the m components on which the portfolio is based. Moreover, when $\mu = \frac{1}{m} \sum_{k=1}^{m} E(x_k)$ or the expected return threshold for
the portfolio is equal to a simple average of the returns of the components on which the portfolio is based then $\lambda_2 = 0$, $\lambda_1 = \frac{2}{m}$ and $\alpha_k = \frac{1}{m}$. This in turn means there will be an equal proportionate investment in the $m$ components comprising the portfolio or that the portfolio will have maximum diversification. Thus when $\mu \neq \frac{1}{m} \sum_{k=1}^{m} E(x_k)$ the portfolio will consist of differing proportionate investments across the asset components on which the portfolio is based.

4. **Momentum**

We have previously observed how there is a steadily growing empirical based literature which argues that a momentum phenomenon influences the evolution of stock market returns; in particular, that equity securities with a short term history of positive abnormal returns will most likely earn positive abnormal returns over the medium term future. Likewise, equity securities with a short term history of negative abnormal returns will most likely incur negative abnormal returns over the medium term future [16,17,18,19]. Here it is important to note that the Schrödinger equation implies that the momentum of a particle moving in the square potential well considered in the previous section (correspondingly, the momentum associated with the return, $x$, on a particular asset) is determined in terms of an operator, $p_x = (\frac{i}{\hbar} d/dx)$, applied to the particle’s wave function, $\psi(x)$ [21,23,37]. One can thus use this momentum operator in conjunction with the wave function (4) to show that the expected or average momentum for the given asset must be:

$$E(p_x) = \int_{-\infty}^{\infty} \psi(x)p_x\psi(x)dx = \frac{1}{i}\int_{-\infty}^{\infty} \psi(x)\psi'(x)dx = 0$$

This means that the expectation value, $E(p_x)$, of the asset’s return momentum is zero – although for any position occupied by the particle within the well corresponding to the return, $x$, the momentum, $p_x\psi(x) = \frac{1}{i}\psi'(x)$, is unlikely to be zero. Likewise, the variance, $\sigma^2(p_x)$, of the asset’s return momentum is given by [23]:
\[
\sigma^2(p_x) = \int_{-\infty}^{\infty} \psi(x)p_x^2\psi(x)dx = -\int_{-\infty}^{\infty} \psi(x)\psi''(x)dx
\]

One can then substitute the wave function (4) into the above expression and thereby show that the variance of the asset’s return momentum will have to be:

\[
\sigma^2(p_x) = C^2 \left[ -\sqrt{\frac{\theta}{\lambda}} \sin(\lambda(b-a)) + \cos(\sqrt{\lambda}(b-a)) \right]^2
\]

\[
+ \frac{(\lambda + \theta)(b-a)}{2} + \frac{(\lambda - \theta)\sin(\sqrt{\lambda}(b-a))\cos(\sqrt{\lambda}(b-a))}{2\sqrt{\lambda}} + \frac{\sqrt{\theta \lambda} \sin^2(\sqrt{\lambda}(b-a))}{\sqrt{\lambda}}
\]

(10)

Now let \( \mu = \int_{-\infty}^{\infty} \psi(x)x\psi(x)dx \) be the expected instantaneous annual return on the given equity security. Moreover, let \( \sigma^2(x) = \int_{-\infty}^{\infty} \psi(x)(x - \mu)^2\psi(x)dx \) be the variance of the instantaneous annual return. Then one can show that the following form of the Heisenberg Uncertainty Principle applies to the relationship between the returns momentum on a particular asset and the return earned by the asset [37]:

\[
\sigma^2(p_x) \sigma^2(x) \geq \frac{1}{4}
\]

(11)

with the equality sign holding if and only if \( \psi^2(x) \) is the Gaussian probability density [15,37]. This result says that the product of the variance associated with the momentum operator, \( p_x \), and the variance of the instantaneous return on the given asset, \( x \), can be no less than \( \frac{1}{4} \). This in turn means that the return on an asset and its momentum are conjugate variables - if the uncertainty associated with the momentum operator declines
then the uncertainty associated with the returns process will have to increase and vice versa.

The parameters summarised in Table 1 were used in conjunction with equation (10) to estimate the variance, \( \sigma^2(p_x) \), of the momentum operator for all 31 components of the FTSE All Share Index. We also used the daily returns for the period from 1994 until 2007 to estimate the variance, \( \sigma^2(x) \), of the annual instantaneous return for each component of the FTSE All Share Index. One can then plot the relationship between \( \sigma^2(p_x) \) and \( \sigma^2(x) \) for all 31 components of the FTSE All Share Index as depicted in Figure 4. Note how this plot generally confirms the prediction of a conjugate relationship between the uncertainty associated with the return momentum and the uncertainty associated with the returns process itself. Moreover, we also have that 

\[
0.3231 \leq \frac{\sigma^2(p_x)}{\sigma^2(x)} \leq 0.901
\]

across the 31 components of the FTSE All Share Index or that the bound defined by equation (12) is satisfied by all thirty one components of the

Index. Finally, our analysis here has the following crucially important implication.

Suppose one approximates the mean instantaneous annual return, \( \mu \), on the equity security by computing the return

\[
\frac{1}{\Delta t} \int_{t-\Delta t}^{t} \frac{P'(s)}{P(s)} \, ds = \frac{1}{\Delta t} \log \left[ \frac{P(t)}{P(t-\Delta t)} \right]
\]

over the interval from time \((t - \Delta t)\) until time \(t\). Then the larger \(\Delta t\) the more likely it is that \(\frac{1}{\Delta t} \log \left[ \frac{P(t)}{P(t-\Delta t)} \right]\) will be a good approximation for the mean instantaneous annual return, \(\mu\). Correspondingly, the smaller \(\Delta t\) the more likely \(\frac{1}{\Delta t} \log \left[ \frac{P(t)}{P(t-\Delta t)} \right]\) will be a poor approximation for \(\mu\). Against this suppose one estimates the instantaneous return’s momentum,

\[
p_x \psi(x(t)) = \frac{1}{i} \psi'(x(t)),
\]

at time \(t\) by using the formula
\[
\frac{1}{i} \frac{\Delta \psi(x(t))}{\Delta x(t)} = \frac{1}{i} \frac{\psi(x(t)) - \psi(x(t - \Delta t))}{x(t) - x(t - \Delta t)} = \frac{1}{i} \psi'(x(t)) + O[(\Delta x(t))^2] = \frac{1}{i} \psi'(x(t)) + O[(\Delta t)^2].
\]

It then follows that \( \frac{1}{i} \frac{\Delta \psi(x(t))}{\Delta x(t)} \) will be a good approximation for \( \frac{1}{i} \psi'(x(t)) \) when \( \Delta t \) is small but a poor approximation for \( \frac{1}{i} \psi'(x(t)) \) when \( \Delta t \) is large. In other words, one can only increase the precision of the measurement of the return’s momentum at the “cost” of less precision in the measurement of the expected return and vice versa.

5. Summary Conclusions

The conventional procedure for assessing the distributional properties of stock market returns uses the Fokker-Planck equation in conjunction with a stochastic differential equation which describes how increments in the returns process evolve over time. This procedure rationalises the returns process in terms of a particle that randomly evolves across the real line. However, a substantial body of empirical evidence shows that models like this lead to problematic descriptions of the way stock market returns evolve through time. Given this, we take the alternative approach of modelling stock market returns in terms of a particle that evolves inside a square potential energy well. The Schrödinger equation is then used to determine the wave function for the returns process. Our empirical analysis shows that the probability distributions implied by this model are strongly compatible with the returns earned by the individual components of the London Stock Exchange’s FTSE All Share Index over the period from 1994 until 2007. We also use the Schrödinger equation to determine the wave functions of investment portfolios and demonstrate how portfolio wave functions based on returns data for the FTSE All Share Index are strongly compatible with the given portfolio’s returns. Finally, we use the definition of returns momentum implied by the Schrödinger equation to show that the uncertainty associated with a share’s return momentum will have an inverse relationship with the uncertainty associated with the returns process itself. Our empirical analysis shows that the predicted inverse relationship between the uncertainty associated with return momentum and the uncertainty associated with the return does in fact hold up for all 31 components of the FTSE All Share Index for which wave functions are estimated in our study. This has the important implication that the return on a share and its momentum are conjugate variables in the sense that one can only increase the precision in the measurement of momentum at the “cost” of less precision in the measurement of the return and vice versa.
Appendix

We wish to minimise the sum of the squared proportionate weights, \( \sum_{k=1}^{m} \alpha_k^2 \), subject to the condition that the proportionate weights sum to unity, \( \sum_{k=1}^{m} \alpha_k = 1 \), and that the portfolio satisfies the expected return threshold, \( \mu = \sum_{k=1}^{m} \alpha_k E(x_k) \). One can thus form the Lagrangian, \( \mathcal{J} \), for this problem [36]:

\[
\mathcal{J} = \sum_{k=1}^{m} \alpha_k^2 + \lambda_1 (1 - \sum_{k=1}^{m} \alpha_k) + \lambda_2 (\mu - \sum_{k=1}^{m} \alpha_k E(x_k)) \quad (A1)
\]

where \( \lambda_1 \) and \( \lambda_2 \) are Lagrange multipliers. The first order condition for \( \alpha_k \) will then be:

\[
\frac{\partial \mathcal{J}}{\partial \alpha_k} = 2 \alpha_k - \lambda_1 - \lambda_2 E(x_k) = 0
\]

or:

\[
\alpha_k = \frac{\lambda_2 E(x_k) + \lambda_1}{2} \quad (A2)
\]

and where \( k = 1, 2, \ldots, m \). Summing across (A2) shows:

\[
\sum_{k=1}^{m} \alpha_k = \frac{\lambda_2}{2} \sum_{k=1}^{m} E(x_k) + \frac{m \lambda_1}{2} = 1 \quad (A3)
\]

from which it follows:

\[
\lambda_1 = \frac{2 - \lambda_2 \sum_{k=1}^{m} E(x_k)}{m} \quad (A4)
\]
One can also multiply through (A3) by $\sum_{k=1}^{m} E(x_k)$ and thereby show:

$$\lambda_2 \frac{1}{m} \left( \sum_{k=1}^{m} E(x_k) \right)^2 + \lambda_1 \sum_{k=1}^{m} E(x_k) = \frac{2}{m} \sum_{k=1}^{m} E(x_k)$$

(A5)

Likewise, multiplying through (A2) by $E(x_k)$ and summing shows:

$$\sum_{k=1}^{m} \alpha_k E(x_k) = \lambda_2 \frac{1}{m} \left( \sum_{k=1}^{m} [E(x_k)]^2 \right) + \lambda_1 \sum_{k=1}^{m} E(x_k) = \mu$$

or equivalently:

$$\lambda_2 \sum_{k=1}^{m} [E(x_k)]^2 + \lambda_1 \sum_{k=1}^{m} E(x_k) = 2\mu$$

(A6)

One can then subtract (A5) from (A6) and thereby show:

$$m \lambda_2 \left( \frac{1}{m} \sum_{k=1}^{m} [E(x_k)]^2 - \frac{1}{m^2} \sum_{k=1}^{m} [E(x_k)]^2 \right) = 2\mu - \frac{2}{m} \sum_{k=1}^{m} E(x_k)$$

or equivalently:

$$\lambda_2 = \frac{2}{m} \cdot \frac{\mu - \frac{1}{m} \sum_{k=1}^{m} E(x_k)}{\frac{1}{m} \sum_{k=1}^{m} [E(x_k)]^2 - \frac{1}{m^2} \left( \sum_{k=1}^{m} E(x_k) \right)^2}$$

(A7)
References


Fig. 1. Motion of a particle representing the return on an asset in a potential energy well. The point marked “a” representing the left hand barrier of the well defines the point at which negative “tunneling” occurs and negative irregular returns are earned by the asset. The point marked “b” representing the right hand barrier of the well defines the point at which positive “tunneling” occurs and positive irregular returns are earned by the asset. Within the well (Region II) regular returns are earned by the asset.
Fig. 2. The upper graph is the estimated wave function, $\psi(x)$, for centred daily returns on the Aerodefence component of the FTSE All Share Index. The estimated parameter values on which the graph is based are $\hat{\lambda} = 0.0484$, $\hat{\gamma} = 0.0229$, $\hat{\theta} = 0.0213$, $\hat{C} = 0.2713$, $\hat{a} = -2.4805$ and $\hat{b} = 2.9188$. The lower but higher peaked graph is the probability density, $\psi^2(x)$, for the daily centred return on the Aerodefence component of the FTSE All Share Index. The lower but less peaked graph is the fitted normal distribution for the Aerodefence component.
Fig. 3. The upper graph is the estimated wave function, $\psi(x)$, for centred daily returns on an equally weighted portfolio comprised of the 31 components of the FTSE All Share Index. The estimated parameter values on which the graph is based are $\hat{\lambda} = 0.1473$, $\hat{\gamma} = 0.0972$, $\hat{\theta} = 0.0531$, $\hat{C} = 0.3701$, $\hat{a} = -1.0285$ and $\hat{b} = 2.1574$. The lower but higher peaked graph is the probability density, $\psi^2(x)$, for the daily centred return on an equally weighted portfolio. The lower but less peaked graph is the fitted normal distribution for the equally weighted portfolio.
Fig. 4. The relationship between the variance, $\sigma^2(p_x)$, of the momentum operator and the variance, $\sigma^2(x)$, of the annualised daily return for 31 components of the FTSE All Share Index based on returns covering the period from 1994 until 2007.
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* significant at the 10% level; ** significant at the 1% level